Quantifying Contextuality

Shane Mansfield*

*Joint work with Samson Abramsky and Rui Soares Barbosa, Oxford University Department of Computer Science

QuPa, IHP, 7th July 2016
Overview

- Unified, general framework for non-locality and contextuality
- Qualitative hierarchy of contextuality
- Quantitative *measure of contextuality*
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- Unified, general framework for non-locality and contextuality
- Qualitative hierarchy of contextuality
- Quantitative *measure of contextuality*

Why?
Overview

• Unified, general framework for non-locality and contextuality
• Qualitative hierarchy of contextuality
• **Quantitative measure of contextuality**

Why?

• Compare degree of contextuality of empirical models
• . . . across different measurement scenarios
• Contextuality as a resource
Contextuality
### Empirical Data (e.g. CHSH)

<table>
<thead>
<tr>
<th>(a, b)</th>
<th>(0, 0)</th>
<th>(0, 1)</th>
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<tr>
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</tbody>
</table>

### Diagram

- $o_A \in \{0, 1\}$
- $o_B \in \{0, 1\}$
- $m_A \in \{a, a'\}$
- $m_B \in \{b, b'\}$
- $p$
Measurement Scenarios: CHSH

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A measurement scenario is a triple \( \langle X, M, O \rangle \) where:
**Measurement Scenarios: CHSH**

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A *measurement scenario* is a triple \(\langle X, \mathcal{M}, O \rangle\) where:

- **\(X\)** a finite set of measurements — e.g.

\[
X = \{a, a', b, b'\}
\]
A measurement scenario is a triple $\langle X, M, O \rangle$ where:

$X$ a finite set of measurements — e.g.

$$X = \{a, a', b, b'\}$$

$M$ the (maximal) contexts — e.g.

$$M = \{\{a, b\}, \{a, b'\}, \{a', b\}, \{a', b'\}\}$$
A measurement scenario is a triple \( \langle X, \mathcal{M}, O \rangle \) where:

- \( X \) a finite set of measurements — e.g. 
  \[
  X = \{a, a', b, b'\}
  \]

- \( \mathcal{M} \) the (maximal) contexts — e.g. 
  \[
  \mathcal{M} = \{\{a, b\}, \{a, b'\}, \{a', b\}, \{a', b'\}\}
  \]

- \( O \) a finite set — e.g. 
  \[
  O = \{0, 1\}
  \]
Measurement Scenarios: ‘Triangle’

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<tr>
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<tr>
<td>(b,c)</td>
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<tr>
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</table>

Measurements:

\[ X = \{a, b, c\} \]

Contexts:

\[ \mathcal{M} = \{\{a, b\}, \{b, c\}, \{c, a\}\} \]

Outcomes:

\[ O = \{0, 1\} \]
### Measurement Scenarios: ‘Triangle’

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**Measurements:**

\[ X = \{a, b, c\} \]

**Contexts:**

\[ \mathcal{M} = \{\{a, b\}, \{b, c\}, \{c, a\}\} \]

**Outcomes:**

\[ O = \{0, 1\} \]
Measurement Scenarios: 18-vector KS

- A set of 18 variables: $X = \{A, \ldots, O\}$
- A set of outcomes: $O = \{0, 1\}$
- A measurement cover: $\mathcal{M} = \{C_1, \ldots, C_9\}$
  whose contexts $C_i$ correspond to the columns in the following table:

<p>| | | | | | | | | |</p>
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Empirical Models

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- Fix a measurement scenario \( \langle X, M, O \rangle \)
Fix a measurement scenario $\langle X, M, O \rangle$

Empirical model: family $\{e_C\}_{C \in M}$ where each $e_C \in \text{Prob}(O^C)$
Empirical Models

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- **Empirical model**: family \(\{e_C\}_{C \in M}\) where each \(e_C \in \text{Prob}(O^C)\)

- Distribution for each context:
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e_{\{a,b\}} = \text{prob}(o_1, o_2 | a, b), \ldots, e_{\{a',b'\}} = \text{prob}(o_1, o_2 | a', b')
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- ‘Local’ consistency:
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Empirical Models

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Contextuality

Classical data should arise as a convex combination of *global assignments*:

\[(a, a', b, b') \mapsto (0, 0, 0, 0), (a, a', b, b') \mapsto (0, 0, 0, 1), \ldots, (a, a', b, b') \mapsto (1, 1, 1, 1)\]

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\[0 \quad 1 \quad b' \quad a' \quad a \quad b\]
Contextuality

Classical data should arise as a convex combination of \textit{global assignments}:

$$(a, a', b, b') \mapsto (0, 0, 0, 0), \quad (a, a', b, b') \mapsto (0, 0, 0, 1), \quad \ldots, \quad (a, a', b, b') \mapsto (1, 1, 1, 1)$$

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### Contextuality

Classical data should arise as a convex combination of **global assignments**:

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Contextuality is present if such a decomposition is not possible
Contextuality

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*Contextuality* is present if such a decomposition is *not* possible

(Contextuality rules out deterministic HVs; non-locality is a special case)
Strong Contextuality

**Strong Contextuality:**

*no* event can be extended to a global assignment.
Strong Contextuality:

**no** event can be extended to a global assignment.

E.g. K–S models, GHZ, the PR box:

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<td>×</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$b_2$</td>
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<td>×</td>
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<td>✓</td>
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</table>
Strong Contextuality:

no event can be extended to a global assignment.

E.g. K–S models, GHZ, the PR box:

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</table>
Contextuality as a Linear System
Contextuality as a Linear System

- Flatten $e$ to a vector $v_e \in \mathbb{R}^m$, e.g.

  $$v_e = \{\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\}$$
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- Similarly for global assignments, e.g.

$$g_1 = \{1, 0, 0, 0, \quad 1, 0, 0, 0, \quad 1, 0, 0, 0, \quad 1, 0, 0, 0\}$$
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- Similarly for global assignments, e.g.

  $$g_1 = \{1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0\}$$

- Define $M := [g_1, \ldots, g_n]$ with global assignments as columns
Contextuality as a Linear System

- Flatten $e$ to a vector $v_e \in \mathbb{R}^m$, e.g.
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  v_e = \{1/2, 0, 0, 1/2, \ 3/8, 1/8, 1/8, 3/8, \ 3/8, 1/8, 1/8, 3/8, \ 1/8, 3/8, 3/8, 1/8\}
  \]

- Similarly for global assignments, e.g.
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- Define $M := [g_1, \ldots, g_n]$ with global assignments as columns

- $e$ is non-contextual iff there exists a solution $d \in \mathbb{R}^m$ with $d \geq 0$ to
  \[
  Md = v_e
  \]
The Contextual Fraction
The Contextual Fraction

Proposition

*Every empirical model admits a convex decomposition*

\[ e = \lambda e^{NC} + (1 - \lambda)e^{SC} \]

*into a non-contextual and a strongly contextual model*
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maximum value \( \lambda \) for such decompositions, denoted \( NC(e) \)
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**Contextual fraction:** \( \text{CF}(e) = 1 - \text{NC}(e) \)

- \( \text{CF}(e) \in [0, 1] \)
- \( e \) is non-contextual iff \( \text{CF}(e) = 0 \)
- \( e \) is strongly contextual iff \( \text{CF}(e) = 1 \)
(Non-)Contextual Fraction via Linear Programming

Checking contextuality of $e$ corresponds to solving

Find $d \in \mathbb{R}^n$

such that $M d = v_e$

and $d \geq 0$
Checking contextuality of $e$ corresponds to solving

Find $d \in \mathbb{R}^n$ such that $M d = v_e$ and $d \geq 0$

Computing the non-contextual fraction corresponds to solving the following linear program:

Find $c \in \mathbb{R}^n$ maximising $1 \cdot c$ subject to $M c \leq v_e$ and $c \geq 0$
Bell Inequality Violations
An inequality for a scenario $\langle X, M, O \rangle$ is given by:

- A set of coefficients $\alpha = \{\alpha(C,s)\}_{C \in M, s \in O}$
- A bound $R$

For a model $e$, $B_\alpha(e) \leq R$, where $B_\alpha(e) = \sum_{C \in M, s \in E(C)} \alpha(C,s) e_{C}(s)$.

Wlog we can take $R$ non-negative (in fact, we can take $R = 0$).

- Bell inequality if it is satisfied by every NC model
- Bell inequality is tight if it is saturated by some NC model
Generalised Bell Inequalities

An inequality for a scenario \( \langle X, \mathcal{M}, O \rangle \) is given by:

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- A bound \( R \)
- For a model \( e \),

\[
\mathcal{B}_\alpha(e) \leq R,
\]

where

\[
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- A vector $\alpha \in \mathbb{R}^m$
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  \[ \alpha \cdot v_e \leq R , \]

  where

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- Bell inequality $\rightarrow$ a bound for $B_\alpha(e)$ amongst NC models
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- Bell inequality $\rightarrow$ a bound for $\mathcal{B}_\alpha(e)$ amongst NC models

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• The **normalised violation** of a Bell inequality $\langle \alpha, R \rangle$ by $e$ is

$$\frac{\max\{0, \mathcal{B}_\alpha(e) - R\}}{\|\alpha\| - R} \in [0, 1]$$
Proposition

Let $e$ be an empirical model

- Normalised violation by $e$ of any Bell inequality is at most $\text{CF}(e)$

- There exists a Bell inequality for which this is attained

- This Bell inequality is tight at “the” non-contextual model $e^{NC}$

$$e = \text{NC}(e) e^{NC} + \text{CF}(e) e^{SC}$$
Contextual Fraction & Bell Violations

Quantifying Contextuality LP:

Find $c \in \mathbb{R}^n$

maximising $1 \cdot c$

subject to $Mc \leq v_e$

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$e = \text{NC}(e) e^{NC} + \text{CF}(e) e^{SC}$

Dual LP:

Find $\mathbf{y} \in \mathbb{R}^m$
minimising $\mathbf{y} \cdot \mathbf{v}_e$
subject to $\mathbf{M}^T \mathbf{y} \geq 1$
and $\mathbf{y} \geq 0$

$\alpha = 1 - |\mathbf{M}|$ 

$\mathbf{M}^T \mathbf{\alpha} \leq 0$
and $\mathbf{\alpha} \leq 1$

computes tight Bell inequality (separating hyperplane)
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- Fully general: applicable to any measurement scenario
- Normalised: allowing comparison across scenarios
- $0$ for non-contextuality, $1$ for strong contextuality
- Computable using linear programming
- Precise relationship to violations of Bell inequalities

What else?
- Computational tools (Mathematica package) implementing all this
- Resource Theory: Monotonicity properties wrt operations that don't introduce contextuality
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1. Calculate quantum empirical models from any (pure or mixed) state and any sets of compatible measurements
2. Calculate the incidence matrix for any measurement scenario
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4. Find the Bell inequality using the dual LP
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1. Equatorial measurements on $|\phi^+\rangle$

- two-qubit Bell state $|\phi^+\rangle = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}$
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- Equatorial measurements at angles $(\phi_1, \phi_2)$
1. Equatorial measurements on $|\phi^+\rangle$

- two-qubit Bell state $|\phi^+\rangle = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}$

- Equatorial measurements at angles $(\phi_1, \phi_2)$

- e.g. $(\phi_1, \phi_2) = (0, \pi/3)$ gives Bell–CHSH model

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\[ \theta = \frac{\pi}{2} \]
1. Equatorial measurements on $|\phi^+\rangle$

Plot $NC(e)$ against measurement angles $(\phi_1, \phi_2)$
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Minima (maximum contextuality since $CF(e) = 1 - NC(e)$):

$$\{\phi_1, \phi_2\} \in \left\{ \left\{ \frac{\pi}{8}, \frac{5\pi}{8} \right\}, \left\{ \frac{7\pi}{8}, \frac{3\pi}{8} \right\} \right\}$$
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$$p = \frac{\sqrt{2} + 2}{8}$$
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$$p = \frac{\sqrt{2} + 2}{8}$$

Note that these achieve Tsirelson violation of the CHSH inequality.
2. Equatorial measurements on GHZ($n$)

Figure: NC($e$) for equatorial measurements at $\phi_1$ and $\phi_2$ on each qubit of $|\psi_{GHZ(n)}\rangle$ with: (a) $n = 3$; (b) $n = 4$. 
Towards a resource theory of contextuality
Contextuality as a resource

- May be more than one useful measure of contextuality
- What properties should a good measure satisfy?
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- Towards a resource theory, as for entanglement (e.g. LOCC), non-locality, ...

- Algebra of empirical models, towards a process calculus?
Operations

- **relabelling**
  \[ e : \langle X, \mathcal{M}, O \rangle, \; \alpha : (X, \mathcal{M}) \cong (X', \mathcal{M}') \rightsquigarrow e[\alpha] : \langle X', \mathcal{M}', O \rangle \]

  For \( C \in \mathcal{M}, s : \alpha(C) \rightarrow O \), \( e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1}) \)
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• restriction
  
  \( e : \langle X, \mathcal{M}, O \rangle, (X', \mathcal{M}') \leq (X, \mathcal{M}) \leadsto e \restriction \mathcal{M}' : \langle X', \mathcal{M}', O \rangle \)

  For \( C' \in \mathcal{M}', s : C' \rightarrow O \), \( (e \restriction \mathcal{M}')_{C'}(s) := e_C|_{C'}(s) \)

  with any \( C \in \mathcal{M} \) s.t. \( C' \subseteq C \)
**Operations**

- **relabelling**
  \[ e : \langle X, M, O \rangle, \ \alpha : (X, M) \cong (X', M') \leadsto e[\alpha] : \langle X', M', O \rangle \]
  
  For \( C \in M, s : \alpha(C) \rightarrow O, e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1}) \)

- **restriction**
  \[ e : \langle X, M, O \rangle, (X', M') \leq (X, M) \leadsto e \upharpoonright M' : \langle X', M', O \rangle \]
  
  For \( C' \in M', s : C' \rightarrow O, (e \upharpoonright M')_{C'}(s) := e_{C'|C'}(s) \)
  
  with any \( C \in M \) s.t. \( C' \subseteq C \)

- **coarse-graining**
  \[ e : \langle X, M, O \rangle, f : O \rightarrow O' \leadsto e/f : \langle X, M, O' \rangle \]
  
  For \( C \in M, s : C \rightarrow O', (e/f)_C(s) := \sum_t : C \rightarrow O, f \circ t = s \ e_C(t) \)
Operations

- **mixing**

  \[ e : \langle X, \mathcal{M}, O \rangle, \quad e' : \langle X, \mathcal{M}, O \rangle, \quad \lambda \in [0, 1] \quad \leadsto \quad e + \lambda e' : \langle X, \mathcal{M}, O \rangle \]

  For \( C \in \mathcal{M} \), \( s : C \rightarrow O' \),

  \[
  (e + \lambda e')_C(s) := \lambda e_C(s) + (1 - \lambda) e'_C(s)
  \]
Operations

- **mixing**
  \[ e : \langle X, M, O \rangle, \ e' : \langle X, M, O \rangle, \lambda \in [0, 1] \leadsto e + \lambda e' : \langle X, M, O \rangle \]

  \[
  \text{For } C \in M, s: C \rightarrow O', \quad (e + \lambda e')_C(s) := \lambda e_C(s) + (1 - \lambda)e'_C(s)
  \]

- **choice**
  \[ e : \langle X, M, O \rangle, \ e' : \langle X', M', O \rangle \leadsto e\&e' : \langle X \sqcup X', M \sqcup M', O \rangle \]

  \[
  \text{For } C \in M, (e\&e')_C := e_C \\
  \text{For } D \in M', (e\&e')_D := e'_D
  \]

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Operations

- **mixing**
  \[ e : \langle X, \mathcal{M}, O \rangle, \ e' : \langle X, \mathcal{M}, O \rangle, \lambda \in [0, 1] \rightsquigarrow e + \lambda \ e' : \langle X, \mathcal{M}, O \rangle \]

  For \( C \in \mathcal{M}, s : C \rightarrow O' \),
  \[ (e + \lambda \ e')_C(s) := \lambda e_C(s) + (1 - \lambda) e'_C(s) \]

- **choice**
  \[ e : \langle X, \mathcal{M}, O \rangle, \ e' : \langle X', \mathcal{M}', O \rangle \rightsquigarrow e \& e' : \langle X \sqcup X', \mathcal{M} \sqcup \mathcal{M}', O \rangle \]

  For \( C \in \mathcal{M}, (e \& e')_C := e_C \)
  For \( D \in \mathcal{M}', (e \& e')_D := e'_D \)

- **tensor**
  \[ e : \langle X, \mathcal{M}, O \rangle, \ e' : \langle X', \mathcal{M}', O \rangle \rightsquigarrow e \otimes e' : \langle X \sqcup X', \mathcal{M} \star \mathcal{M}', O \rangle \]

  \[ \mathcal{M} \star \mathcal{M}' := \{ C \sqcup D \mid C \in \mathcal{M}, D \in \mathcal{M}' \} \]
  For \( C \in \mathcal{M}, D \in \mathcal{M}', s = \langle s_1, s_2 \rangle : C \sqcup D \rightarrow O, \)
  \[ (e \otimes e')_{C \sqcup D}(s_1, s_2) := e_C(s_1) e'_D(s_2) \]
Operations and the Contextual Fraction

• relabelling
  \[ \text{CF}(e[\alpha]) = \text{CF}(e) \]

• restriction
  \[ \text{CF}(e \upharpoonright \sigma') \leq \text{CF}(e) \]

• coarse-graining
  \[ \text{CF}(e/f) \leq \text{CF}(e) \]

• mixing
  \[ \text{CF}(e + \lambda e') \leq \lambda \text{CF}(e) + (1 - \lambda) \text{CF}(e') \]

• choice
  \[ \text{CF}(e & e') = \max\{\text{CF}(e), \text{CF}(e')\} \]

\[ \text{NCF}(e & e') = \min\{\text{NCF}(e), \text{NCF}(e')\} \]

• tensor
  \[ \text{CF}(e_1 \otimes e_2) = \text{CF}(e_1) + \text{CF}(e_2) - \text{CF}(e_1) \text{CF}(e_2) \]

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- **tensor**
  \[ \text{CF}(e_1 \otimes e_2) = \]
  \[ \text{CF}(e_1) + \text{CF}(e_2) - \text{CF}(e_1)\text{CF}(e_2) \]
  \[ \text{NCF}(e_1 \otimes e_2) = \text{NCF}(e_1)\text{NCF}(e_2) \]
Further directions

- Alternative measures: e.g. Negative Probabilities
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- Signalling models
  - Empirical data will not always satisfy no-signalling

\[ e_{\text{NS}} = \lambda e_{\text{NS}} - (1 - \lambda) e_{\text{SS}} \]

- Analysis of real data:
  
  \[ e_{\text{Delft}} \approx 0.0664 e_{\text{SS}} + 0.4073 e_{\text{SC}} + 0.5263 e_{\text{NC}} \]

  \[ e_{\text{NIST}} \approx 0.0000049 e_{\text{SS}} + 0.0000281 e_{\text{SC}} + 0.9999670 e_{\text{NC}} \]

- First extract NS fraction, then NC fraction? Or vice-versa?

- Non-uniqueness of witnesses!

- Connections with Contextuality-by-Default (Dzhafarov et al.) and Markham & Winter

- Resource Theory

- Sequencing. . .

- What is this resource useful for?
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- Signalling models
  - Empirical data will not always satisfy no-signalling
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- Similarly, a no-signalling fraction:
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Conclusion

*Introducing a measure of contextuality... the Contextual Fraction*

- Fully general: applicable to any measurement scenario
- Normalised: allowing comparison across scenarios
- 0 for non-contextuality ... 1 for strong contextuality
- Computable using linear programming
- Precise relationship to *violations of Bell inequalities*
- Computational tools (*Mathematica* package) implementing all this
- **Resource Theory**: Monotonicity properties wrt operations that don’t introduce contextuality
Questions...