Quantifying Contextuality

Shane Mansfield*

*Joint work with Samson Abramsky and Rui Soares Barbosa,
Oxford University Department of Computer Science

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Overview

- Unified, general framework for non-locality and contextuality
- Qualitative hierarchy of contextuality
- **Quantitative *measure of contextuality***

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- Qualitative hierarchy of contextuality
- **Quantitative *measure of contextuality***

Why?

- Compare degree of contextuality of empirical models
- ...across different measurement scenarios
- Contextuality as a resource

Contextuality

Empirical Data (e.g. CHSH)

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
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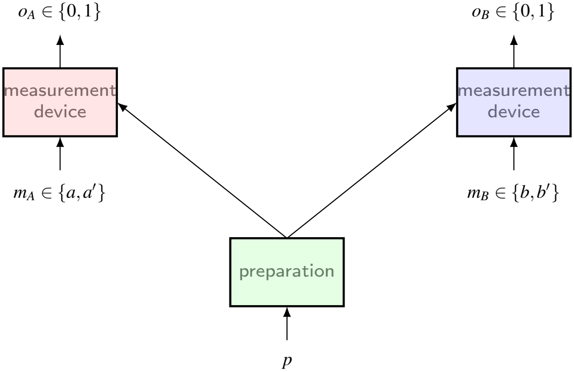

Measurement Scenarios: CHSH

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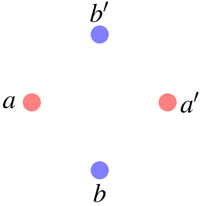

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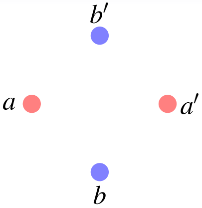

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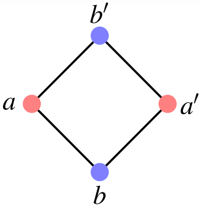

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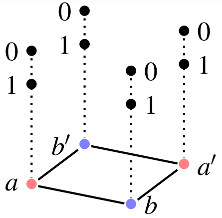

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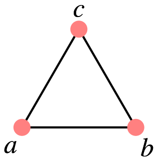
O a finite set — e.g.

$$O = \{0,1\}$$

Measurement Scenarios: 'Triangle'

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	0	$1/2$	$1/2$	0
(b,c)	0	$1/2$	$1/2$	0
(c,a)	0	$1/2$	$1/2$	0



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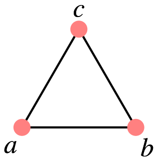

Measurements:

$$X = \{a, b, c\}$$

Contexts:

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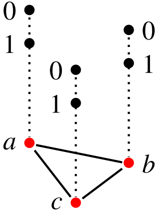

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Measurement Scenarios: 18-vector KS

- A set of 18 variables: $X = \{A, \ldots, O\}$
- A set of outcomes: $O = \{0, 1\}$
- A measurement cover: $\mathscr{M} = \{C_1, \ldots, C_9\}$
  whose contexts C_i correspond to the columns in the following table:

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A	A	H	H	B	I	P	P	Q
B	E	I	K	E	K	Q	R	R
C	F	C	G	M	N	D	F	M
D	G	J	L	N	O	J	L	O

Empirical Models

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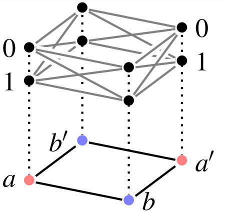

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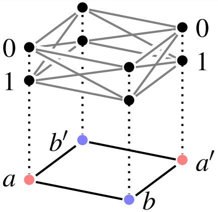

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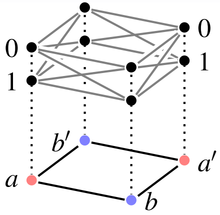

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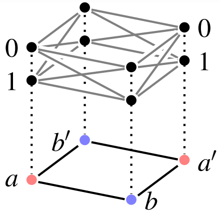

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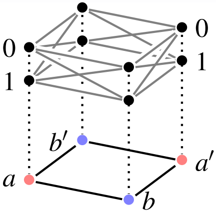

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NO-SIGNALLING

Contextuality

Classical data should arise as a convex combination of *global assignments*:

$(a,a',b,b') \mapsto (0,0,0,0)$, $(a,a',b,b') \mapsto (0,0,0,1)$, $\ldots$,
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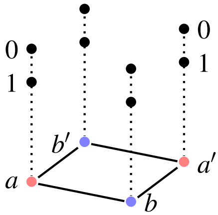

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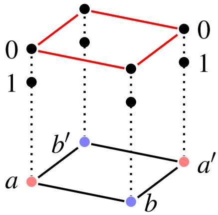

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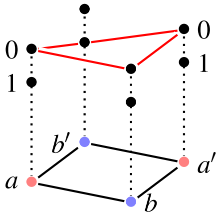

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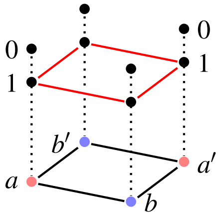

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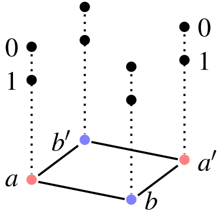

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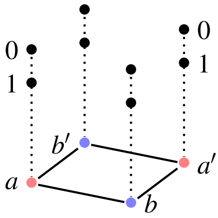

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(Contextuality rules out deterministic HVs; non-locality is a special case)

Strong Contextuality

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E.g. K–S models, GHZ, the PR box:

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a_1	b_2	✓	×	×	✓
a_2	b_1	✓	×	×	✓
a_2	b_2	×	✓	✓	×

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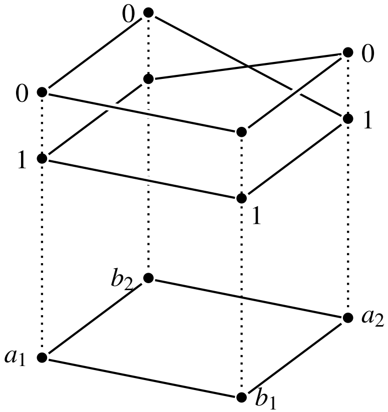

Contextuality as a Linear System

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- Flatten e to a vector $\mathbf{v}_e \in \mathbb{R}^m$, e.g.

$$\mathbf{v}_e = \{1/2, 0, 0, 1/2, \quad 3/8, 1/8, 1/8, 3/8, \quad 3/8, 1/8, 1/8, 3/8, \quad 1/8, 3/8, 3/8, 1/8\}$$

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$$\mathbf{g}_1 = \{1, 0, 0, 0, \quad 1, 0, 0, 0, \quad 1, 0, 0, 0, \quad 1, 0, 0, 0\}$$

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- e is *non-contextual* iff there exists a solution $\mathbf{d} \in \mathbb{R}^m$ with $\mathbf{d} \geq 0$ to

$$\mathbf{M}\mathbf{d} = \mathbf{v}_e$$

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Proposition

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Contextual fraction: $\mathsf{CF}(e) = 1 - \mathsf{NC}(e)$

- $\mathsf{CF}(e) \in [0,1]$
- e is *non-contextual* iff $\mathsf{CF}(e) = 0$
- e is *strongly contextual* iff $\mathsf{CF}(e) = 1$

(Non-)Contextual Fraction via Linear Programming

Checking contextuality of e corresponds to solving

$$\begin{array}{ll} \texttt{Find} & \mathbf{d} \in \mathbb{R}^n \\ \texttt{such that} & \mathbf{M}\mathbf{d} = \mathbf{v}_e \\ \texttt{and} & \mathbf{d} \geq \mathbf{0} \end{array}$$

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Computing the non-contextual fraction corresponds to solving the following *linear program*:

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Bell Inequality Violations

Generalised Bell Inequalities

An **inequality** for a scenario $\langle X, \mathscr{M}, O \rangle$ is given by:

- A set of coefficients $\alpha = \{\alpha_{(C,s)}\}_{C \in \mathscr{M}, s \in O^C}$
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where

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- **Bell inequality** if it is satisfied by every NC model
- Bell inequality is **tight** if it is saturated by some NC model

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- The **normalised violation** of a Bell inequality $\langle \alpha, R \rangle$ by e is

$$\frac{\max\{0, \mathscr{B}_\alpha(e) - R\}}{\|\alpha\| - R} \quad \in [0,1]$$

Contextual Fraction & Bell Violations

Proposition

Let e be an empirical model

- *Normalised violation by e of any Bell inequality is at most* $\mathsf{CF}(e)$
- *There exists a Bell inequality for which this is attained*
- *This Bell inequality is tight at "the" non-contextual model e^{NC}*

$$e = \mathsf{NC}(e)\, e^{NC} + \mathsf{CF}(e)\, e^{SC}$$

Contextual Fraction & Bell Violations

Quantifying Contextuality LP:

`Find`	$\mathbf{c} \in \mathbb{R}^n$
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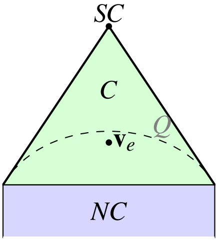

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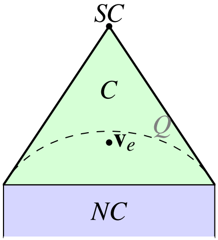

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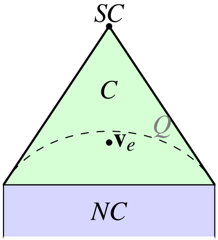

Dual LP:

Find	$\mathbf{y} \in \mathbb{R}^m$
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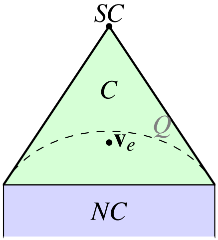

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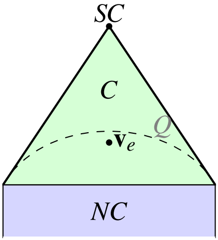

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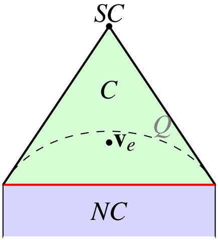

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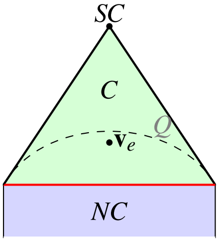

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4. Find the Bell inequality using the dual LP

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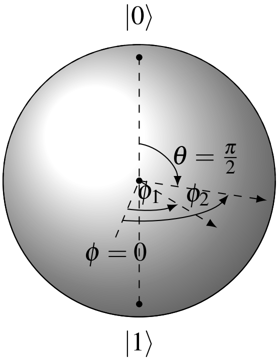

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- Equatorial measurements at angles (ϕ_1, ϕ_2)
- e.g. $(\phi_1, \phi_2) = (0, \pi/3)$ gives Bell–CHSH model

A	B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
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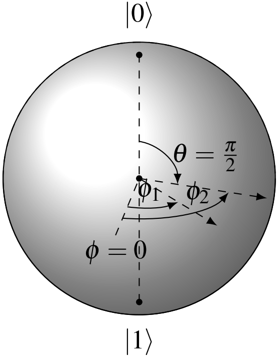

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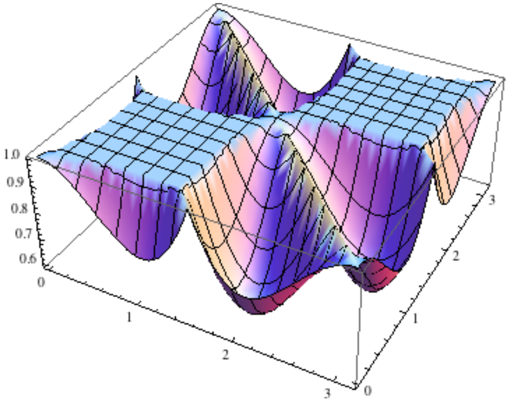

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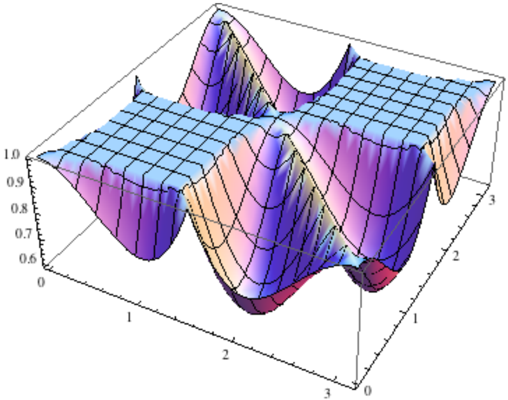

Plot $\mathsf{NC}(e)$ against measurement angles (ϕ_1, ϕ_2)

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$$\{\phi_1, \phi_2\} \in \left\{ \left\{ \frac{\pi}{8}, \frac{5\pi}{8} \right\}, \left\{ \frac{7\pi}{8}, \frac{3\pi}{8} \right\} \right\}$$

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Note that these achieve Tsirelson violation of the CHSH inequality.

2. Equatorial measurements on GHZ(n)

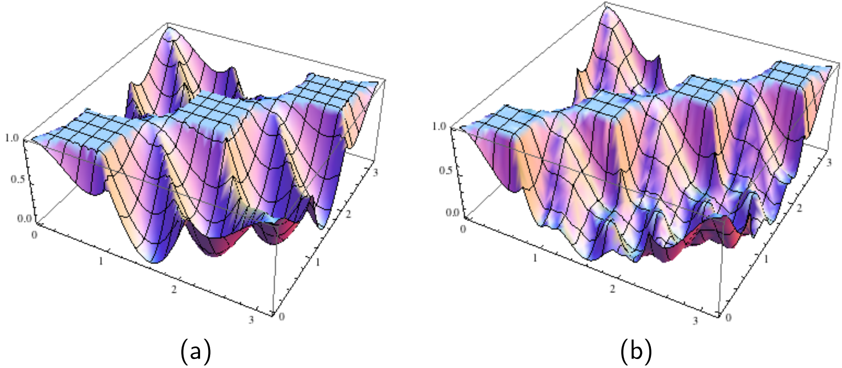

Figure: NC(e) for equatorial measurements at ϕ_1 and ϕ_2 on each qubit of $\left|\psi_{\mathrm{GHZ}(n)}\right\rangle$ with: (a) $n = 3$; (b) $n = 4$.

Towards a resource theory of contextuality

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- Algebra of empirical models, towards a process calculus?

Operations

- *relabelling*
  $e : \langle X, \mathscr{M}, O \rangle,\ \alpha : (X, \mathscr{M}) \cong (X', \mathscr{M}')\ \rightsquigarrow\ e[\alpha] : \langle X', \mathscr{M}', O \rangle$

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- *restriction*
  $e : \langle X, \mathscr{M}, O\rangle,\ (X', \mathscr{M}') \leq (X, \mathscr{M}) \ \rightsquigarrow\ e \restriction \mathscr{M}' : \langle X', \mathscr{M}', O\rangle$

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- *coarse-graining*
  $e : \langle X, \mathscr{M}, O \rangle,\ f \colon O \longrightarrow O' \ \rightsquigarrow\ e/f : \langle X, \mathscr{M}, O' \rangle$

  For $C \in M, s \colon C \longrightarrow O'$, $(e/f)_C(s) := \sum_{t \colon C \longrightarrow O, f \circ t = s} e_C(t)$

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  $e : \langle X, \mathscr{M}, O\rangle,\ e' : \langle X, \mathscr{M}, O\rangle, \lambda \in [0,1] \ \rightsquigarrow\ e +_\lambda e' : \langle X, \mathscr{M}, O\rangle$

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- *tensor*

  $e : \langle X, \mathscr{M}, O\rangle,\ e' : \langle X', \mathscr{M}', O\rangle \ \rightsquigarrow\ e \otimes e' : \langle X \sqcup X', \mathscr{M} \star \mathscr{M}', O\rangle$

  $\mathscr{M} \star \mathscr{M}' := \{C \sqcup D \mid C \in \mathscr{M}, D \in \mathscr{M}'\}$
  For $C \in \mathscr{M}, D \in \mathscr{M}', s = \langle s_1, s_2\rangle : C \sqcup D \longrightarrow O$,
  $$(e \otimes e')_{C \sqcup D}\langle s_1, s_2\rangle := e_C(s_1)\, e'_D(s_2)$$

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  - Sequencing...
  - What is this resource useful for?

Conclusion

Introducing a measure of contextuality... **the Contextual Fraction**

- Fully general: applicable to any measurement scenario
- Normalised: allowing comparison across scenarios
- 0 for non-contextuality $\ldots 1$ for strong contextuality
- Computable using linear programming
- Precise relationship to *violations of Bell inequalities*
- Computational tools (*Mathematica* package) implementing all this
- **Resource Theory:** Monotonicity properties wrt operations that don't introduce contextuality

Questions...

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